Received: *July 20, 2000*, Accepted: *October 18, 2000*

$O(d,d)$-invariant collapse/inflation from colliding superstring waves

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Abstract: Generalizing previous work, we study the collision of massless superstring plane waves in D space-time dimensions within an explicitly $O(D-2, D-2)$-invariant set of field equations. We discuss some general properties of the solutions, showing in particular that they always lead to the formation of a singularity in the future. Using the above symmetry, we obtain entire classes of new analytic solutions with non-trivial metric, dilaton and antisymmetric field, and discuss some of their properties of specific relevance to string cosmology.

Keywords: Bosonic Strings, String Duality, Physics of the Early Universe.

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JHEP10(2000)035

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1. Introduction

One of the most debated issues in pre-big bang (PBB) cosmology [1] concerns its initial conditions [2, 3]. How fine-tuned should these be in order to generate a Universe resembling ours? The principle of asymptotic past triviality (APT) [3] amounts to saying that, far enough into the past, the Universe was well described by a *generic*, perturbative solution of the low-energy, tree-level string effective action. In other words, the very early Universe is assumed to lie deeply inside the low-curvature, small-coupling regime. Under this assumption, for a critical superstring theory admitting the trivial Minkowski vacuum order by order in perturbation theory, the early Universe can be represented as a random superposition (a chaotic sea) of massless waves, propagating in all directions and with all (sub-string scale) frequencies. The particle content of the waves should represent *all* the massless degrees of freedom of superstring theory, i.e. *each one* of the possible marginal deformations of the two-dimensional CFT in trivial space-time. This can be hardly called a fine-tuned initial state!

Understanding the evolution of such a Universe is clearly *not* a simple task. However, we know that, as long as the tree-level low-energy approximation remains valid, the classical field dynamics is both scale- and dilaton-shift-invariant since the string and Planck scales simply sit as overall factors in front of the action. General arguments suggest that, while most of the time these waves will just propagate linearly and independently, occasionally, through positive interference, overdense regions will form and, thanks to gravitational instability, will lead to gravitational collapse. It was argued [3] that the interiors of sufficiently large and weakly coupled collapsing regions could give birth to Universes that could resemble our own. The above-mentioned classical symmetries guarantee that the scale of collapse will itself be a stochastic variable, whose distribution will be related to that of the original distribution of wave lengths. Since such a distribution naturally contains arbitrarily long (though not arbitrarily short) wavelengths, it looks very likely that large enough black holes would form with non-vanishing probability.

Notwithstanding the appeal of these general arguments, it would be very instructive to see them at work in some explicitly soluble model. In [3] this was partly done, in $D = 4$ for the spherically symmetric case, through use of some powerful results obtained by Christodoulou [4] over more than a decade. More recently, exact analytic solutions were constructed by Feinstein, Kunze and Vázquez-Mozo (hereafter FKV) [5], who replaced the chaotic sea of waves by two colliding, homogeneous, planar-fronted waves. In [5] the waves have infinite fronts and thus always lead to collapse on space-time scales L that are inversely related to the energy density (i.e. energy per unit of transverse area) in the waves:

$$L \sim (G_N \rho_1 G_N \rho_2)^{-1/2} . \tag{1.1}$$

While FKV only deal with $D = 4$, we will show that the above result holds for any D. In the following we shall denote by $d = D - 2$ the number of transverse coordinates.

Although the case of infinite fronts is obviously an idealization, causality arguments lead to the conclusion that collapse takes place even for finite-front waves, provided the typical transverse extension R of the waves is larger than L, i.e. if:

$$R > L \sim (G_N \rho_1 G_N \rho_2)^{-1/2} = R^d (G_N E_1 G_N E_2)^{-1/2} = \frac{R^d}{G_N \sqrt{s}} = R_s \left(\frac{R}{R_s}\right)^d , \tag{1.2}$$

i.e. provided $R_s \equiv (G_N \sqrt{s})^{\frac{1}{d-1}} > R$, as naively expected.

Other limitations of the FKV paper are that it deals only with gravitational and dilatonic waves and that, as we mentioned, it is restricted to $D = 4$. It has been pointed out recently [6] that the approach to the cosmological singularity (which appears as the $r = 0$ singularity inside the collapsing region) depends strongly both on the dimensionality of space and on the presence of other fields, in particular of the various p-forms that string theory possesses. It was argued, in particular, that turning on all the forms present in any consistent string (or M) theory, changes the monotonic Kasner-like behaviour of the gravi-dilaton system into an ever-oscillating behaviour à la BKL [7].

As a first step in the direction of overcoming the two limitations of the work by FKV, we shall extend it both to an arbitrary number of (non-compact) dimensions and to the presence of the Kalb-Ramond $B_{\mu\nu}$ field. Keeping the exact planar symmetry allows (see section 2) for an explicitly $O(d, d)$-invariant formulation of the dynamics in the string frame. While several properties of the solutions can be discussed in the general case (section 3), we have only been able, so far, to construct explicit analytic solutions in the case of $B_{\mu\nu} = 0$ and arbitrary dimensions (sections 4 and 5), or when the $B_{\mu\nu}$ field is generated through $O(d,d)$-transformations (section 6). This will suffice, however, to address some of the issues raised in ref. [6].

2. The $O(d,d)$-invariant field equations

As explained in the introduction, we start with the low-energy, tree-level effective action of critical superstring theory in the string frame. Up to a classically irrelevant overall factor

$$S = \int dx^D \sqrt{-g} e^{-\phi} \left(\mathrm{R} + \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \tag{2.1}$$

where ϕ is the dilaton, R is the curvature scalar for the metric tensor $g_{\mu\nu}$, and $H_{\mu\nu\rho}$ is the field strength for the antisymmetric field

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \,. \tag{2.2}$$

The problem we wish to study is the head-on collision of two infinite plane-front waves, which, without loss of generality, will be taken to move along the x^1 axis. Assuming symmetry under translations along the $d = D - 2$ spatial directions orthogonal to x^1 (x^i with $i = 2, \ldots, D-1$), it is obvious that the problem is endowed with d abelian isometries. In a convenient coordinate frame, there will be no dependence upon the "transverse" coordinates x^i, and thus, according to general arguments [9, 10], we expect to have an exact $O(d,d)$ invariance of the classical field equations. The symmetry of the field equations (not to be confused with a true quantum symmetry) is most easily exhibited in terms of the invariance of a "reduced action" living in the non-trivial (here two-dimensional) subspace. In the case of $O(D-1, D-1)$-symmetry (homogeneous Bianchi-I cosmologies) this was done in [9, 11] and led to a continuous extension of scale-factor duality of Bianchi-I cosmologies [12].

For the present purposes, we will adapt to our case a general result by Maharana and Schwarz [13], and write the reduced action coming from (2.1) as:

$$S = \int dx^0 dx^1 \sqrt{-g} e^{-\overline{\phi}} \left[R + \partial_\alpha \overline{\phi} \partial^\alpha \overline{\phi} + \frac{1}{8} \operatorname{Tr} \left(\partial_\alpha M^{-1} \partial^\alpha M \right) \right], \tag{2.3}$$

whose notation we shall now explain.

The equations of motion allow the metric $g_{\mu\nu}$ to be taken block-diagonal, with blocks given by g_{ij} and $g_{\alpha\beta}$ where roman indices $(i, j, \ldots)$ will span the components of the tensors from 2 to $d+1$ while the indices α and β take the values 0 and 1. The explicit metric and curvatures appearing in (2.3) only refer to the latter.

We arrange the components g_{ij} in a d-dimensional matrix G. The matrix B will contain the components B_{ij} of the antisymmetric field while the remaining components of B are set to 0. M is then the $2d$-dimensional matrix defined by:

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} . \tag{2.4}$$

Finally, the shifted dilaton is defined by:

$$\overline{\phi} = \phi - \frac{1}{2} \log \det G + \text{constant} \, , \tag{2.5}$$

where the constant will be conveniently fixed later.

The reduced action (2.3) is manifestly invariant under the transformations

$$\begin{aligned} g_{\alpha\beta} &\longrightarrow g_{\alpha\beta} \, , \\ \overline{\phi} &\longrightarrow \overline{\phi} \, , \\ M &\longrightarrow \Omega^T M \Omega \, , \end{aligned} \tag{2.6}$$

where Ω denotes a global $O(d,d)$ transformation

$$\Omega^T \eta \Omega = \eta \, , \tag{2.7}$$

with η the $O(d,d)$ metric in off-diagonal form

$$\eta = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} . \tag{2.8}$$

M itself belongs to $O(d,d)$ and, being symmetric, satisfies

$$M \eta M = \eta \, , \qquad \text{i.e. } M^{-1} = \eta M \eta \, . \tag{2.9}$$

The above equations make the check of $O(d,d)$-invariance trivial.

Let us now derive the (manifestly $O(d,d)$-invariant) field equations from (2.3). Varying the action with respect to the shifted dilaton gives

$$R + 2g^{\alpha\beta} D_\alpha D_\beta \overline{\phi} - g^{\alpha\beta} \partial_\alpha \overline{\phi} \partial_\beta \overline{\phi} + \frac{1}{8} g^{\alpha\beta} \operatorname{Tr} \left(\partial_\alpha M^{-1} \partial_\beta M \right) = 0 \, , \tag{2.10}$$

while varying with respect to the 2-metric $g_{\alpha\beta}$ provides

$$R_{\alpha\beta} + D_\alpha D_\beta \overline{\phi} + \frac{1}{8} \operatorname{Tr} \left(\partial_\alpha M^{-1} \partial_\beta M \right) = 0 \, . \tag{2.11}$$

Combining the trace of the latter equation with the previous one gives a simple equation for $\overline{\phi}$ which will play an important role later:

$$D^\alpha D_\alpha \exp(-\overline{\phi}) = 0 \, . \tag{2.12}$$

The variation with respect to M must be performed carefully because of the constraints coming from its definition. Following [9], we take them into account by writing

$$\delta M = \Omega^T M \Omega - M \,, \tag{2.13}$$

with $\Omega = 1 + \epsilon$ in $O(d,d)$. Expanding to first order in ϵ, we obtain

$$\partial_\alpha \left(e^{-\overline{\phi}} \sqrt{-g} g^{\alpha\beta} M^{-1} \partial_\beta M \right) = 0 \,. \tag{2.14}$$

Equations (2.10), (2.11), (2.12) and (2.14) compose the full set we wish to solve. Before doing so, let us write them more explicitly by going to the conformal gauge for the 2-metric

$$g_{\alpha\beta} = e^F \eta_{\alpha\beta} \tag{2.15}$$

and by working with "light-cone" coordinates:

$$u = \frac{x^0 - x^1}{\sqrt{2}} \,, \qquad v = \frac{x^0 + x^1}{\sqrt{2}} \,. \tag{2.16}$$

Equations (2.10), (2.11), (2.12), and (2.14) then simply become:

$$\partial_u \partial_v \exp(-\overline{\phi}) = 0 \,, \qquad \text{i.e. } \partial_u \partial_v \overline{\phi} = \partial_u \overline{\phi} \partial_v \overline{\phi} \,; \tag{2.17}$$

$$\partial_u \left(e^{-\overline{\phi}} M^{-1} \partial_v M \right) + \partial_v \left(e^{-\overline{\phi}} M^{-1} \partial_u M \right) = 0 \,; \tag{2.18}$$

$$\partial_u^2 \overline{\phi} - \partial_u F \partial_u \overline{\phi} + \frac{1}{8} \mathrm{Tr} \left(\partial_u M^{-1} \partial_u M \right) = 0 \,, \qquad \text{same with } u \to v \,; \tag{2.19}$$

$$\partial_u \partial_v \overline{\phi} - \partial_u \partial_v F + \frac{1}{8} \mathrm{Tr} \left(\partial_u M^{-1} \partial_v M \right) = 0 \,. \tag{2.20}$$

Note that the two equations (2.19) have the form of Virasoro constraints of a two-dimensional CFT, while the other equations are of the evolution type. Furthermore, a straightforward, though not completely trivial, calculation shows that the integrability condition for eqs. (2.19) holds and that eq. (2.20) is just a consequence of the previous ones. Our discussion will thus be based on solving the set of eqs. (2.17)–(2.19).

3. General properties of the solutions

In this section we will derive some general features of the solutions, which will be useful for the more detailed investigations to be described later on.

The two colliding waves are defined to have their fronts at $u = 0$ and $v = 0$, respectively, and thus to collide at $u = v = 0$ (i.e. at $x^0 = x^1 = 0$). The two waves are *not* assumed to be impulsive, i.e. their energy density can have any (finite?) support at positive u and v, respectively. Space-time is thus naturally divided in four regions:

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Region I, defined by $u, v < 0$, is the space-time in front of the waves before any interaction takes place. It is trivial Minkowski space-time:

$$ds_I^2 = -2dudv + \sum \left(dx^i\right)^2 , \qquad B = 0 , \qquad \phi = \phi_0 , \tag{3.1}$$

with a constant perturbative dilaton ($\exp(\phi_0) \ll 1$). It will be convenient to fix the constant in (2.5) as $-\phi_0$ so that, in region I, $\overline{\phi} = 0$.

Region II, defined by $u > 0$, $v < 0$, is the wave coming from the left before the interaction. Metric and dilaton depend only on u and therefore the field equations allow us to take $F = 0$ and write the ansatz:

$$ds_{\mathrm{II}}^2 = -2dudv + G_{ij}^{\mathrm{II}}(u)dx^i dx^j , \qquad B_{ij} = B_{ij}^{\mathrm{II}}(u) , \qquad \phi = \phi^{\mathrm{II}}(u) . \tag{3.2}$$

Similarly, region III, defined by $u < 0$, $v > 0$, represents the wave coming from the right before the interaction. There,

$$ds_{\mathrm{III}}^2 = -2dudv + G_{ij}^{\mathrm{III}}(v)dx^i dx^j , \qquad B_{ij} = B_{ij}^{\mathrm{III}}(v) , \qquad \phi = \phi^{\mathrm{III}}(v) . \tag{3.3}$$

Note that we have not assumed any special shape for G, so that the results we give in this section will hold whatever the (relative) polarization of the waves.

Finally, region IV ($u > 0$, $v > 0$) is the interaction region, with

$$ds_{\mathrm{IV}}^2 = -2e^F dudv + G_{ij}^{\mathrm{IV}} dx^i dx^j , \tag{3.4}$$

and F, G^{IV}, ϕ^{IV} and B^{IV} are all functions of both u and v. Of course, the metric must be continuous along with its derivative on the boundary lines between the four regions. The same must be true for the dilaton ϕ and the antisymmetric field B.

Let us begin by solving the equations in region II (and thus, by trivial analogy, in region III). The only non-trivial equation is the "Virasoro constraint", eq. (2.19), which (after momentarily dropping the subscripts II) reads

$$\ddot{\overline{\phi}} = \frac{1}{8} \mathrm{Tr} \left[\left(M^{-1} \dot{M} \right)^2 \right] , \tag{3.5}$$

where the dot indicates the derivative with respect to u. The r.h.s. can be written as the sum of three terms:

$$\frac{1}{8} \mathrm{Tr} \left[\left(M^{-1} \dot{M} \right)^2 \right] = \frac{1}{4d} \left[\mathrm{Tr} \left(G^{-1} \dot{G} \right) \right]^2 + \frac{1}{4} \left\{ \mathrm{Tr} \left[\left(G^{-1} \dot{G} \right)_t^2 \right] - \right.$$
$$\left. - \mathrm{Tr} \left[\left(G^{-1} \dot{B} \right)^2 \right] \right\} , \tag{3.6}$$

where $(G^{-1}\dot{G})_t$ is the traceless part of $(G^{-1}\dot{G})$. The first trace on the r.h.s. can be expressed in terms of the dilaton and the shifted dilaton as

$$\frac{1}{4d} \left[\mathrm{Tr} \left(G^{-1} \dot{G} \right) \right]^2 = \frac{1}{4d} \left(\frac{d}{du} \log \det G \right)^2 = \frac{1}{d} \left(\dot{\phi} - \dot{\overline{\phi}} \right)^2 . \tag{3.7}$$

It is now useful to change variable from u to $\tilde{u}$, with

$$\frac{d}{du} = e^{-2\phi/d}\frac{d}{d\tilde{u}}\,. \tag{3.8}$$

Then, eq. (3.5) becomes

$$e^{\overline{\phi}/d}\left(e^{-\overline{\phi}/d}\right)'' = -\frac{1}{d^2}\phi'^2 - \frac{1}{4d}\left\{ \mathrm{Tr}\left[\left(G^{-1}G'\right)^2_t\right] - \mathrm{Tr}\left[\left(G^{-1}B'\right)^2\right]\right\}, \tag{3.9}$$

where the prime denotes the derivative with respect to $\tilde{u}$.

It can be easily proved that all terms on the r.h.s. of eq. (3.9) are negative definite, if we remember that G is symmetric with positive eigenvalues and B is antisymmetric. Hence, for any non-trivial wave, $e^{-\overline{\phi}/d}$, which is constant and identically equal to 1 in region I, must acquire a non-vanishing, negative, and never increasing derivative in region II. Thus, $e^{-\overline{\phi}/d}$ must vanish at some finite $\tilde{u} = \tilde{u}^*$. Returning now to the coordinate u, we see that, if the dilaton is bounded (a necessary assumption if we want to use the tree-level effective action), there exists a finite $u = u^*$ where $e^{-\overline{\phi}/d}$ vanishes. Correspondingly, also $\det G$ vanishes, and the metric of the transverse space will collapse to zero proper volume, thereby producing a (coordinate) singularity.

It is not too difficult to estimate the order of magnitude of u^* by multiplying eq. (3.9) by $e^{-\overline{\phi}/d}$ and by integrating it once after having gone back to the original u-variable. The result is an estimate of u^{*-1} and is given in terms of an integral over u of the energy density per unit volume, i.e. in terms of the energy density per unit area. One thus recovers the estimate $u^* \sim (G_N\rho)^{-1}$, in which the appearance of Newton's constant is somewhat fictitious since, for waves of a given geometry, the energy density scales like G_N^{-1}. The final result, reported in eq. (1.1), is a Lorentz-boost-invariant way of writing the same expression once both waves are considered simultaneously.

The same arguments can be repeated in region III, where $\det G$ has to vanish at some finite $v = v^*$ with $\overline{\phi} \to +\infty$. These results generalize to any D and to non-trivial antisymmetric fields, a well-known result in $D = 4$.

Let us finally analyse region IV, where the interaction between the two waves occurs. We drop the subscript IV from all functions. We begin by using eq. (2.17), which tells us that $e^{-\overline{\phi}}$ is the sum of a function of u and a function of v. The unique function of this type that matches the boundary conditions with region I is

$$e^{-\overline{\phi}(u,v)} = e^{-\overline{\phi}_{\mathrm{II}}(u)} + e^{-\overline{\phi}_{\mathrm{III}}(v)} - 1\,. \tag{3.10}$$

We see that $e^{-\overline{\phi}(u,v)}$ must vanish on a hypersurface joining the coordinate singularities in regions II and III and contained in the region $u \leq u_0$, $v \leq v_0$ within region IV.

Let us now introduce two new sets of coordinates that simplify the analysis in region IV. One set is of the light-cone type:

$$r = r(v) = 2e^{-\overline{\phi}_{\mathrm{III}}(v)} - 1\,, \qquad s = s(u) = 2e^{-\overline{\phi}_{\mathrm{II}}(u)} - 1\,, \tag{3.11}$$

while the second set is of the $t - x$ kind:

$$\xi = \frac{1}{2}(r+s) = e^{-\overline{\phi}(u,v)} \sim -t\,,$$
$$z = \frac{1}{2}(s-r) = e^{-\overline{\phi}_{\rm II}(u)} - e^{-\overline{\phi}_{\rm III}(v)}\,. \tag{3.12}$$

Note that the coordinates r, s run from $+1$ to -1 in region IV with their sum always positive except on the singular boundary where $r + s = 0$. Going from the original coordinates to either of the new sets changes only the conformal factor of the 2-metric. We may thus write, for instance,

$$ds^2_{\rm IV} = -e^f d\xi^2 + e^f dz^2 + G_{ij} dx^i dx^j\,, \tag{3.13}$$

where f and G are functions of ξ and z. In r, s coordinates, this becomes

$$ds^2_{\rm IV} = -2e^f dr ds + G_{ij} dx^i dx^j\,, \tag{3.14}$$

where f and G are now functions of r and s. The shifted dilaton is simply

$$\overline{\phi} = -\log\xi = -\log\frac{(r+s)}{2}\,. \tag{3.15}$$

Finally, in these coordinates, eq. (2.17) becomes trivial and the only equations to be solved, (2.18) and (2.19), become

$$\partial_r \left((r+s)M^{-1}\partial_s M\right) + \partial_s \left((r+s)M^{-1}\partial_r M\right) = 0\,, \tag{3.16}$$
$$(r+s)^{-2} + (r+s)^{-1}\partial_r F + \frac{1}{8}\,{\rm Tr}\left(\partial_r M^{-1}\partial_r M\right) = 0\,, \tag{3.17}$$
$$(r+s)^{-2} + (r+s)^{-1}\partial_s F + \frac{1}{8}\,{\rm Tr}\left(\partial_s M^{-1}\partial_s M\right) = 0\,. \tag{3.18}$$

To end this section, let us discuss the relation between the above equations and those discussed in refs. [9, 14], where general solutions in the homogeneous [9] and quasi-homogeneous [14] case were derived. Here we are dealing with fields depending also on one spatial coordinate. For this reason, we have built the matrix M by considering only the remaining d spatial dimensions, excluding the g_{11} component. In the approach to the singularity, we expect the time derivatives to dominate over the spatial ones (this hypothesis can also be tested and verified *a posteriori*, see section 6). Thus, close to the singularity, the general solutions of [9, 14] should be recovered. To see the explicit connection between the equations of motion (3.16) and (3.18) and those in [9, 14], we have to complete the M matrix with the missing rows and columns, extending it to a $2(d+1)$-dimensional matrix. We must also add the g_{11} component in the determinant of the metric used to construct the shifted dilaton and go over to cosmic time $dT = e^{f/2}d\xi$. After some simple algebra, the equations of motion can be brought to a form identical to that of refs. [9, 14], strongly suggesting that the asymptotic solutions of [14] will emerge near the singularity.

4. Solutions with $B = 0$

The case of $B = 0$ with parallel polarized waves was discussed in ref. [5] for $d = 2$. We shall obtain below a generalization of their results to arbitrary d. For $B = 0$, M reduces to

$$M = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} . \tag{4.1}$$

In each region, we choose a diagonal G with $G_{ii} = e^{\lambda+\psi_i}$, where $\sum \psi_i = 0$ and $\lambda = \frac{1}{d} \log \det G$ goes to $-\infty$ at $u = u^*$ in region II and at $v = v^*$ in region III.

The equations of motion follow easily from eqs. (3.16) and (3.18). They read

$$\lambda_{,rs} + \frac{1}{2\,(r+s)}\,(\lambda_{,r} + \lambda_{,s}) = 0\,, \tag{4.2}$$

$$\psi_{i,rs} + \frac{1}{2\,(r+s)}\,(\psi_{i,r} + \psi_{i,s}) = 0\,, \tag{4.3}$$

$$f_{,r} + \frac{1}{(r+s)} - \frac{r+s}{4}\left(d\lambda_{,r}^2 + \sum \psi_{i,r}^2\right) = 0\,, \tag{4.4}$$

$$f_{,s} + \frac{1}{(r+s)} - \frac{r+s}{4}\left(d\lambda_{,s}^2 + \sum \psi_{i,s}^2\right) = 0\,. \tag{4.5}$$

We see that the equations for the ψ_i and that for λ are decoupled and can be solved separately. They are also formally the same, so the solutions, found by Szekeres [8], have the same structure:

$$\begin{aligned} -(r+s)^{1/2}\psi_i\,(r,s) = & \int_s^1 ds' \left[(1+s')^{1/2}\psi_i(1,s')\right]_{,s'} P_{-1/2}\left[1 + 2\frac{(1-r)\,(s'-s)}{(1+s')\,(r+s)}\right] + \\ & + \int_r^1 dr' \left[(1+r')^{1/2}\psi_i(r',1)\right]_{,r'} \times \\ & \times P_{-1/2}\left[1 + 2\frac{(1-s)\,(r'-r)}{(1+r')\,(r+s)}\right] . \end{aligned} \tag{4.6}$$

The same expression holds for λ, with the obvious replacements. In the above expressions, $P_{-1/2}(x)$ are Legendre functions written in standard notation. Once λ is given, ϕ can be obtained from eq. (2.5), since $\overline{\phi}$ is known. Finally, the function $f(r,s)$ is given by an integral along a curve joining the point $(r = s = 1)$, where it vanishes, to the generic point (r,s):

$$f\,(r,s) = \int_{(1,1)}^{(r,s)} [f_{,r}dr + f_{,s}ds,] \tag{4.7}$$

with $f_{,r}$ and $f_{,s}$ given by the r.h.s. of eqs. (4.4) and (4.5) as functions of (r,s) through the previously determined ϕ and ψ_i.

We see that λ and the ψ_i are singular on the hypersurface $\xi = r + s = 0$. So, in a very general way, we find that the collision of two plane waves leads to a (curvature) singularity in the space-time whatever the number of dimensions. Although we do not have, at the moment, such an explicit solution for the general case, the discussion given in the previous section makes us believe that a curvature singularity will always emerge along the hypersurface $\xi = 0$.

5. Asymptotic approach to the singularity

For the solutions (4.6), the asymptotic behaviour for $\xi \to 0$ is easily found by taking the large-argument limit of the Legendre function [8] (see also Yurtsever [15]):

$$\begin{aligned} \psi_i(\xi, z) &\sim \epsilon_i(z) \log \xi \,, \\ \lambda(\xi, z) &\sim \kappa(z) \log \xi \,, \\ f(\xi, z) &\sim a(z) \log \xi \,. \end{aligned} \tag{5.1}$$

The coefficients multiplying the logarithm are functions of z, whose range, on the singular surface, is $-1 - +1$. One easily finds

$$\begin{aligned} \epsilon_i(z) = {} & \frac{1}{\pi\sqrt{1+z}} \int_z^1 ds \left[(1+s)^{1/2} \psi(1,s) \right]_{,s} \left(\frac{s+1}{s-z} \right)^{1/2} + \\ & + \frac{1}{\pi\sqrt{1-z}} \int_{-z}^1 dr \left[(1+r)^{1/2} \psi(r,1) \right]_{,r} \left(\frac{r+1}{r+z} \right)^{1/2} , \end{aligned} \tag{5.2}$$

$$a(z) = \frac{1}{4} \sum \epsilon_i^2(z) + \frac{d}{4} \kappa^2(z) - 1 \,, \tag{5.3}$$

with $\kappa(z)$ given by the same expression as ϵ_i with ψ_i replaced by λ. The sum of the ϵ_i must be zero according to the definition of the ψ_i.

The asymptotic form of the metric is

$$ds^2_{\mathrm{IV}} = -\xi^{a(z)} d\xi^2 + \xi^{a(z)} dz^2 + \xi^{\kappa(z)} \sum \xi^{\epsilon_i(z)} \left(dx^i \right)^2 , \tag{5.4}$$

while

$$\phi \sim -\left(1 + \frac{d}{2} \kappa(z) \right) \log \xi \,. \tag{5.5}$$

Going over to cosmic time $\xi = t^{\frac{2}{a(z)+2}}$, gives the metric in Kasner form with exponents

$$\begin{aligned} p_1(z) &= \frac{a(z)}{a(z)+2} \,, \\ p_i(z) &= \frac{\kappa(z) + \epsilon_i(z)}{a(z)+2} \,. \end{aligned} \tag{5.6}$$

The following relations are immediately verified:

$$\begin{aligned} \phi &= \left(\sum_{\alpha=1}^{D-1} p_\alpha(z) - 1 \right) \log t \,, \\ \sum_{\alpha=1}^{D-1} & p_\alpha^2(z) = 1 \,. \end{aligned} \tag{5.7}$$

The behaviour of the fields near the singularity is thus of Kasner type, modified, as usual, by the presence of the dilaton. Note that, at the two tips of the singularity, ϵ_i and κ diverge in such a way that the Kasner exponents near the tips are simply $p_1 = 1$, $p_i = 0$. This corresponds to a (contracting) Milne-like metric which, being non-singular, nicely matches the non-singular behaviour in regions II and III. Away from these two points, κ and ϵ_i can take any value: it is easy to verify that the whole Kasner sphere can be covered by appropriately choosing the initial data.

For generic z, the collision leads to a singularity showing all the characteristics of a cosmological singularity. The Kasner exponents depend on z and therefore the behaviour of the metric will depend on the point of the singular hypersurface that we approach as $\xi \to 0$. It is therefore interesting to study the signs of the Kasner exponents for different choices of the coefficients ϵ_i and κ. In particular, we would like to see whether, far from the tips of the singularity, inflation may take place.

By the definition of $a(z)$, we see that the denominators of the Kasner exponents (5.6) are always positive. Concerning the numerators, we note that $a(z)$ is a quadratic form in ϵ_i and κ. The equation $a(z) = 0$ thus defines an ellipsoid in the space $(\epsilon_i; \kappa)$ with p_1 positive outside of this ellipsoid and negative inside. For each i the equation $p_i = 0$ defines the plane $\kappa + \epsilon_i = 0$ passing through the centre of the former ellipsoid. The p_i's are positive on the semispace containing the positive κ semiaxis. Finally, we have to remember the constraint on the sum of the ψ_i, which becomes $\sum \epsilon_i = 0$ in the asymptotic regime.

In conclusion, we see that there indeed exists a region where all the Kasner exponents are negative. In fact, if we stay inside the ellipsoid on the side where $\kappa < 0$, and take all ϵ_i close enough to zero, all the exponents are negative and we have inflation in all directions. In the opposite situation (i.e. outside the ellipsoid, with $\kappa > 0$ and ϵ_i close to zero) all exponents are positive and we have contraction in all directions. All intermediate cases are also possible as we can let any number of ϵ_i be large and negative and take at least one ϵ_i large enough and positive to balance the others.

6. Colliding waves with antisymmetric field

Although we have written the equations in a manifestly $O(d,d)$-covariant form, we have only managed, so far, to find solutions in region IV for vanishing antisymmetric field. We can perform, however, $O(d,d)$ boosts on our solutions to introduce it. This procedure can be worked out in any number of the dimensions. Our purpose in this section is to discuss the main features of the cosmologies obtained by this procedure and to clarify the role of the antisymmetric field, particularly in connection with the work of ref. [6].

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Let us write the transformation matrix Ω in terms of four d-dimensional blocks

$$\Omega = \begin{pmatrix} P & R \\ Q & S \end{pmatrix} . \tag{6.1}$$

The constraint (2.7) reads

$$\begin{aligned} P^T Q + Q^T P &= 0 \,, \qquad (6.2)\\ R^T Q + S^T P &= I_d \,, \\ P^T S + Q^T R &= I_d \,, \\ R^T S + S^T R &= 0 \,. \qquad (6.3) \end{aligned}$$

From eq. (2.13), with M in the form (4.1), we obtain the inverse of the new metric G'

$$G'^{-1} = P^T G^{-1} P + Q^T G Q \tag{6.4}$$

and extract also the antisymmetric field

$$B' = \left(R^T G^{-1} P + S^T G S \right) G' \,. \tag{6.5}$$

It is interesting to investigate the asymptotic behaviour of the new cosmologies obtained this way. We try to be very general, considering an initial metric that is diagonal and has the Kasner form

$$G = \mathrm{diag} \left(t^{\lambda_1}, \ldots , t^{\lambda_d} \right) , \tag{6.6}$$

with the λ_i ordered so that $i < j$ implies that $|\lambda_i| \geq |\lambda_j|$. In section 5 we showed that the asymptotic approach to the singularity of the colliding waves metric is of this type, if we let the λ_i depend on z.

When we perform the $O(d,d)$ transformation, the new metric G' is no longer diagonal. However, we can diagonalize it by a step-by-step procedure. Exploiting the fact that G has the form (6.6), the eigenvalue equation for G'^{-1} takes the following asymptotic form:

$$\sum_b \sum_i \left(P_{ia} t^{-\lambda_i} P_{ib} + Q_{ia} t^{\lambda_i} Q_{ib} \right) v_b = \alpha v_a \,. \tag{6.7}$$

Assuming, in a first instance, that all λ_i are non-vanishing, we define two new matrices U and U', whose components are

$$\begin{aligned} U_{ij} &= \begin{cases} P_{ij} & \text{if } \lambda_i > 0 \\ Q_{ij} & \text{if } \lambda_i < 0 \,, \end{cases} \\ U'_{ij} &= \begin{cases} Q_{ij} & \text{if } \lambda_i > 0 \\ P_{ij} & \text{if } \lambda_i < 0 \,. \end{cases} \end{aligned} \tag{6.8}$$

The dominant term in the eigenvalue equation as $t \to 0$ becomes

$$\sum_b U_{1a} t^{-|\lambda_1|} U_{1b} v_b = \alpha v_a \,. \tag{6.9}$$

This is the eigenvalue equation for the projector to U_1, i.e. the first eigenvalue of G'^{-1} is $\alpha = |U_1|^2 t^{-|\lambda_1|}$ with eigenvector $v_a = U_{1a}$. Since the vectors orthogonal to U_1 annihilate the dominant term we just considered, in order to find the other eigenvalues and the corresponding eigenvectors, we consider the successive sub-dominant terms in the original eigenvalue equation in the space orthogonal to U_1.

The next term is

$$\sum_b U_{2a} t^{-|\lambda_2|} U_{2b} v_b = \alpha v_a \,. \tag{6.10}$$

The eigenvector of this term is $v = U_2$: however, in order to annihilate the dominant term, we have to take the component of U_2 (asymptotically) orthogonal to U_1 as the second eigenvector. The corresponding eigenvalue is $\alpha = (|U_2|^2 - \frac{(U_1 \cdot U_2)^2}{|U_1|^2}) t^{-|\lambda_2|}$. The next eigenvector must now be taken in the space orthogonal to U_1 and U_2, and so on. In this way, we can build the complete system of eigenvectors of G'^{-1}, combining the rows of U, which are rows of the two matrices P and Q. The d eigenvalues are then proportional to $t^{-|\lambda_i|}$ and therefore they are all diverging as $t \to 0$. It is easy to extend this discussion to the case of some vanishing λ_i. This clearly implies asymptotically vanishing eigenvalues for G.

In order to complete our discussion, we have to mention some possible exceptions to the above-mentioned situation. Not always, the rows of U, used to build the eigenvectors, form an independent set of vectors. For a particular class of $O(d,d)$ transformations, having $\det U = 0$, the result is different. It is easy to understand the modifications: each time we face a U_i linearly dependent on the previous rows, we just ignore it and proceed with the algorithm to the next term in G'^{-1}. When we have exhausted the first d dominant terms, we will still have to find one eigenvalue (having skipped one term in the diagonalization). Taking the next term of G'^{-1}, in the form

$$U'_{da} t^{|\lambda_d|} U'_{db} \,, \tag{6.11}$$

we can find the missing eigenvalue. In conclusion, we can see that, if the rank of U is $s \leq d$, then G'^{-1} will have s diverging and $d-s$ vanishing eigenvalues. Note that, even if the matrix P is chosen freely, the matrix Q is related to P by eq. (6.2). We have to check whether, because of this relation, the determinant of U is necessarily vanishing for some initial values of the Kasner exponents.

First, we observe that (6.2) implies

$$\det P \det Q = (-1)^d \det Q \det P \,. \tag{6.12}$$

Therefore, for odd d, either the determinant of P or the determinant of Q vanish. We may now distinguish two classes of $O(d,d)$ transformations. The identity transformation belongs to those giving a vanishing $\det Q$, and the same is true for all transformations that can be reduced to the identity in a continuous way. For metrics that are inflating in all directions before the transformation, $U = Q$ and the transformations having $\det Q = 0$, instead of turning all eigenvalues to contracting ones, leave at least an inflating. Conversely, for metrics contracting in all dimensions, $U = P$ and transformations with $\det P = 0$ cannot leave all the eigenvalues unchanged, but induce inflation in at least one direction.

Consider, as the last peculiar case, the one in which the exponents in the initial metric are all positive except one. Then it is possible to show that the constraint (6.2) forces $\det U = 0$ if P is non-singular. The same argument can be repeated for metrics inflating in all directions except one. In this case, U is composed by $d-1$ rows from Q and one from P. If Q is non-singular, then $\det U = 0$ and again we have one inflating dimension left.

Summing up, we can say that cosmologies with antisymmetric field that can be obtained, by an $O(d,d)$ transformation, from a metric having Kasner behaviour near the singularity, generally contract in all dimensions. In even dimensions, at least one inflates when the original metric has just one inflating or just one contracting dimension. In odd dimensions, if the $O(d,d)$ transformation has $\det Q = 0$, then, starting from a full inflating metric or from a metric with one inflating dimension, we are left with one inflating dimension. If $\det P = 0$, then one dimension inflates when we start from a full contracting metric or from a metric having just one contracting dimension.

We should also recall that none of the $O(d,d)$ transformations affects the g_{11} component of the metric, which, therefore, can either inflate or contract.

The results of ref. [16] are compatible with ours, since, for $d = 2$, our statements, when the determinant of G inflates, can be summarized as a change of sign in λ while ψ is left invariant.

7. Conclusions

In this paper, extending previous work [5], we have modelled the onset of pre-big bang inflation, from asymptotically trivial initial conditions [3], as the result of the collision of two plane waves, made of gravitons, dilatons, and Kalb-Ramond massless particles, in any number $D = d + 2$ of space-time dimensions. We showed that the evolution of the system is described in terms of a compact and elegant set of $O(d,d)$-invariant equations and that properties of the solutions can be studied in full generality in three of the four regions defined by the planar-collision problem. This already enables us to argue that the formation of a curvature singularity in the future (to be identified with the big bang) is generic.

However, so far, we have not been able to solve the general problem analytically in the fourth, and most interesting, region, except for the case of dilatonic and parallel-polarized gravitational waves. Nonetheless, using $O(d,d)$ transformations, we were able to construct new solutions containing the KR field and to discuss the

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physically relevant properties of these new solutions. The main conclusions that appear to emerge are the following:

- While the formation of a singularity is generic, the existence of inflating regions near the singular surface is only generic in the absence of the KR form.
- The KR form, at least when generated from $O(d,d)$ transformations, tends to generate contraction rather than expansion, in agreement with other results [1] and arguments [6].
- Since our equations appear to reduce, near the singularity, to those studied previously [9, 14] in the homogeneous (or quasi-homogeneous) case, it looks very likely that the above behaviour will persist in the general solution.

Eventually, one may be led to the conclusion that the most generic APT initial conditions (i.e. those containing all possible kinds of waves in the initial state) can hardly produce such a flat, homogeneous and isotropic Universe to dispense us completely from the more standard kind of potential-energy-driven post-big bang inflation. Actually, it is very likely that, after exit from pre-big bang inflation, the dilaton and other moduli will find themselves displaced from the minima of their non-perturbatively-generated potentials and that further inflation will result from their rolling down towards them. A similar conclusion seems to follow from completely different, more phenomenological arguments, i.e. from the recent data analysis of CMB anisotropies at small angular scales [17], which appears to confirm the need for adiabatic perturbations of the kind naturally provided by potential-driven inflation, but absent in the PBB scenario.

If so, we will have to back up from the early claims that the PBB scenario can replace altogether standard inflation, and settle instead for a complementary role it would play in providing the initial conditions that standard inflation badly needs, and in "explaining", from more natural and generic initial conditions, the most mysterious event in the entire life of our Universe, the big bang.

Note added. While this paper was being written, we received a new paper by Damour and Henneaux [18] containing a discussion of the outcome of $O(d,d)$ transformations on Kasner-like solutions. Their results, obtained with a different diagonalization procedure, agree with those discussed in our section 6.

Acknowledgments

We are grateful to Maurizio Gasperini for help during the early stage of this work and to Thibault Damour for discussions concerning his recent work, refs. [6, 18]. V.B. was supported by European Social Fund through PO of MURST. He also wishes to thank the Theory Division at CERN for hospitality while this work was carried out.

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